

EXPONENTIAL SUMS ON \mathbf{A}^n

ALAN ADOLPHSON AND STEVEN SPERBER

ABSTRACT. We discuss exponential sums on affine space from the point of view of Dwork's p -adic cohomology theory.

1. INTRODUCTION

Let p be a prime number, $q = p^a$, \mathbf{F}_q the finite field of q elements. Associated to a polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$ and a nontrivial additive character $\Psi : \mathbf{F}_q \rightarrow \mathbf{C}^\times$ are exponential sums

$$(1.1) \quad S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) = \sum_{x_1, \dots, x_n \in \mathbf{F}_{q^i}} \Psi(\text{Trace}_{\mathbf{F}_{q^i}/\mathbf{F}_q} f(x_1, \dots, x_n))$$

and an L -function

$$(1.2) \quad L(\mathbf{A}^n, f; t) = \exp\left(\sum_{i=1}^{\infty} S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) \frac{t^i}{i}\right).$$

Let d = degree of f and write

$$f = f^{(d)} + f^{(d-1)} + \dots + f^{(0)},$$

where $f^{(j)}$ is homogeneous of degree j . A by now classical theorem of Deligne[2, Théorème 8.4] says that if $(p, d) = 1$ and $f^{(d)} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-1} , then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$, all of whose reciprocal roots have absolute value equal to $q^{n/2}$. This implies the estimate

$$(1.3) \quad |S(\mathbf{A}^n(\mathbf{F}_{q^i}), f)| \leq (d-1)^n q^{ni/2}.$$

In this article, we give a p -adic proof of the fact that $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$ (equation (2.14) and Theorem 3.8) and give p -adic estimates for its reciprocal roots, namely, we find a lower bound for the p -adic Newton polygon of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ (Theorem 4.3). Using general results of Deligne[3], this information can be used to compute l -adic cohomology and hence again obtain the archimedean estimate (1.3) (Theorem 5.3).

For Theorems 3.8 and 4.3, we need to assume only that $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$ (or, equivalently, that $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ have no common zero in \mathbf{P}^{n-1}). When $(p, d) = 1$, this is equivalent to Deligne's hypothesis. When d is divisible by p , there are only a few cases satisfying this regular sequence condition. We check them by hand in section 6 to prove the following slight generalization of Deligne's result.

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Theorem 1.4. *Suppose $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$. Then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$, all of whose reciprocal roots have absolute value equal to $q^{n/2}$.*

In the article [1], we dealt with exponential sums on tori. After a general coordinate change, one can, by using the standard toric decomposition of \mathbf{A}^n , deduce most of the results of this article from results in [1]. Our main purpose here is to develop some new methods that will be more widely applicable. For instance, recent results of García[6] on exponential sums on \mathbf{A}^n do not seem to follow from [1].

In contrast with [1], we work systematically with spaces of type $C(b)$ (convergent series on a closed disk) and avoid spaces of type $L(b)$ (bounded series on an open disk). This ties together more closely the calculation of p -adic cohomology and the estimation of the Newton polygon of the characteristic polynomial of Frobenius, eliminating much of section 3 of [1].

Another new feature of this work is the use of the spectral sequence associated to the filtration by p -divisibility on the complex $\Omega_{C(b)}$ (section 3 below). Although the behavior of this spectral sequence is rather simple in the setting of this article (namely, $E_1^{r,s} = E_\infty^{r,s}$ for all r and s), we believe it will play a significant role in more general situations, such as that of García[6]. We hope the methods developed here will allow us to extend the results of this article to those situations.

2. PRELIMINARIES

In this section, we review the results from Dwork's p -adic cohomology theory that will be used in this paper.

Let \mathbf{Q}_p be the field of p -adic numbers, ζ_p a primitive p -th root of unity, and $\Omega_1 = \mathbf{Q}_p(\zeta_p)$. The field Ω_1 is a totally ramified extension of \mathbf{Q}_p of degree $p-1$. Let K be the unramified extension of \mathbf{Q}_p of degree a . Set $\Omega_0 = K(\zeta_p)$. The Frobenius automorphism $x \mapsto x^p$ of $\text{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\text{Gal}(\Omega_0/\Omega_1) (\simeq \text{Gal}(K/\mathbf{Q}_p))$ by requiring $\tau(\zeta_p) = \zeta_p$. Let Ω be the completion of an algebraic closure of Ω_0 . Denote by “ord” the additive valuation on Ω normalized by $\text{ord } p = 1$ and by “ord _{q} ” the additive valuation normalized by $\text{ord}_q q = 1$.

Let $E(t)$ be the Artin-Hasse exponential series:

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

Let $\gamma \in \Omega_1$ be a solution of $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $\text{ord } \gamma = 1/(p-1)$ and consider

$$(2.1) \quad \theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Omega_1[[t]].$$

The series $\theta(t)$ is a splitting function in Dwork's terminology[4]. Furthermore, its coefficients satisfy

$$(2.2) \quad \text{ord } \lambda_i \geq i/(p-1).$$

We consider the following spaces of p -adic functions. Let b be a positive rational number and choose a positive integer M such that Mb/p and $Md/(p-1)$ are integers. Let π be such that

$$(2.3) \quad \pi^{Md} = p$$

and put $\tilde{\Omega}_1 = \Omega_1(\pi)$, $\tilde{\Omega}_0 = \Omega_0(\pi)$. The element π is a uniformizing parameter for the rings of integers of $\tilde{\Omega}_1$ and $\tilde{\Omega}_0$. We extend $\tau \in \text{Gal}(\Omega_0/\Omega_1)$ to a generator of $\text{Gal}(\tilde{\Omega}_0/\tilde{\Omega}_1)$ by requiring $\tau(\pi) = \pi$. For $u = (u_1, \dots, u_n) \in \mathbf{R}^n$, we put $|u| = u_1 + \dots + u_n$. Define

$$(2.4) \quad C(b) = \left\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \mid A_u \in \tilde{\Omega}_0 \text{ and } A_u \rightarrow 0 \text{ as } |u| \rightarrow \infty \right\}.$$

For $\xi = \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b)$, define

$$\text{ord } \xi = \min_{u \in \mathbf{N}^n} \{\text{ord } A_u\}.$$

Given $c \in \mathbf{R}$, we put

$$C(b, c) = \{\xi \in C(b) \mid \text{ord } \xi \geq c\}.$$

Let $\hat{f} = \sum_u \hat{a}_u x^u \in K[x_1, \dots, x_n]$ be the Teichmüller lifting of the polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$, i. e., $(\hat{a}_u)^q = \hat{a}_u$ and the reduction of \hat{f} modulo p is f . Set

$$(2.5) \quad F(x) = \prod_u \theta(\hat{a}_u x^u),$$

$$(2.6) \quad F_0(x) = \prod_{i=0}^{a-1} \prod_u \theta((\hat{a}_u x^u)^{p^i}).$$

The estimate (2.2) implies that $F \in C(b, 0)$ for all $b < 1/(p-1)$ and $F_0 \in C(b, 0)$ for all $b < p/q(p-1)$. Define an operator ψ on formal power series by

$$(2.7) \quad \psi\left(\sum_{u \in \mathbf{N}^n} A_u x^u\right) = \sum_{u \in \mathbf{N}^n} A_{pu} x^u.$$

It is clear that $\psi(C(b, c)) \subseteq C(pb, c)$. For $0 < b < p/(p-1)$, let $\alpha = \psi^a \circ F_0$ be the composition

$$C(b) \hookrightarrow C(b/q) \xrightarrow{F_0} C(b/q) \xrightarrow{\psi^a} C(b).$$

Then α is a completely continuous $\tilde{\Omega}_0$ -linear endomorphism of $C(b)$. We shall also need to consider $\beta = \tau^{-1} \circ \psi \circ F$, which is a completely continuous $\tilde{\Omega}_1$ -linear (or $\tilde{\Omega}_0$ -semilinear) endomorphism of $C(b)$. Note that $\alpha = \beta^a$.

Set $\hat{f}_i = \partial \hat{f} / \partial x_i$ and let $\gamma_l = \sum_{i=0}^l \gamma^{p^i} / p^i$. By the definition of γ , we have

$$(2.8) \quad \text{ord } \gamma_l \geq \frac{p^{l+1}}{p-1} - l - 1.$$

For $i = 1, \dots, n$, define differential operators D_i by

$$(2.9) \quad D_i = \frac{\partial}{\partial x_i} + H_i,$$

where

$$(2.10) \quad H_i = \sum_{l=0}^{\infty} \gamma_l p^l x_i^{p^l - 1} \hat{f}_i^{p^l}(x^{p^l}) \in C\left(b, \frac{1}{p-1} - b \frac{d-1}{d}\right)$$

for $b < p/(p-1)$. Thus D_i and “multiplication by H_i ” operate on $C(b)$ for $b < p/(p-1)$.

To understand the definition of the D_i , put

$$\begin{aligned}\hat{\theta}(t) &= \prod_{i=0}^{\infty} \theta(t^{p^i}), \\ \hat{F}(x) &= \prod_u \hat{\theta}(\hat{a}_u x^u),\end{aligned}$$

so that

$$\begin{aligned}F(x) &= \hat{F}(x)/\hat{F}(x^p), \\ F_0(x) &= \hat{F}(x)/\hat{F}(x^q).\end{aligned}$$

Then formally

$$\begin{aligned}\alpha &= \hat{F}(x)^{-1} \circ \psi^a \circ \hat{F}(x) \\ \beta &= \hat{F}(x)^{-1} \circ \tau^{-1} \circ \psi \circ \hat{F}(x).\end{aligned}$$

It is trivial to check that $x_i \partial / \partial x_i$ and ψ commute up to a factor of p , hence the differential operators

$$\hat{F}^{-1} \circ x_i \frac{\partial}{\partial x_i} \circ \hat{F} = x_i \frac{\partial}{\partial x_i} + \frac{x_i \partial \hat{F} / \partial x_i}{\hat{F}}$$

formally commute with α (up to a factor of q) and β (up to a factor of p). From the definitions, one gets

$$\hat{\theta}(t) = \exp\left(\sum_{l=0}^{\infty} \gamma_l t^{p^l}\right).$$

It then follows that

$$\frac{x_i \partial \hat{F} / \partial x_i}{\hat{F}} = x_i H_i,$$

which gives

$$(2.11) \quad \alpha \circ x_i D_i = q x_i D_i \circ \alpha,$$

$$(2.12) \quad \beta \circ x_i D_i = p x_i D_i \circ \beta.$$

Consider the de Rham-type complex $(\Omega_{C(b)}, D)$, where

$$\Omega_{C(b)}^k = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} C(b) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and $D : \Omega_{C(b)}^k \rightarrow \Omega_{C(b)}^{k+1}$ is defined by

$$D(\xi dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \left(\sum_{i=1}^n D_i(\xi) dx_i \right) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

We extend the mapping α to a mapping $\alpha : \Omega_{C(b)} \rightarrow \Omega_{C(b)}$ defined by linearity and the formula

$$\alpha_k(\xi dx_{i_1} \wedge \dots \wedge dx_{i_k}) = q^{n-k} \frac{1}{x_{i_1} \dots x_{i_k}} \alpha(x_{i_1} \dots x_{i_k} \xi) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Equation (2.11) implies that α_\cdot is a map of complexes. The Dwork trace formula, as formulated by Robba[8], then gives

$$(2.13) \quad L(\mathbf{A}^n/\mathbf{F}_q, f; t) = \prod_{k=0}^n \det(I - t\alpha_k \mid \Omega_{C(b)}^k)^{(-1)^{k+1}}.$$

From results of Serre[9] we then get

$$(2.14) \quad L(\mathbf{A}^n/\mathbf{F}_q, f; t) = \prod_{k=0}^n \det(I - t\alpha_k \mid H^k(\Omega_{C(b)}, D))^{(-1)^{k+1}},$$

where we denote the induced map on cohomology by α_k also.

3. FILTRATION BY p -DIVISIBILITY

The p -adic Banach space $C(b)$ has a decreasing filtration $\{F^r C(b)\}_{r=-\infty}^\infty$ defined by setting

$$F^r C(b) = \left\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b) \mid A_u \in \pi^r \mathcal{O}_{\tilde{\Omega}_0} \text{ for all } u \right\},$$

where $\mathcal{O}_{\tilde{\Omega}_0}$ denotes the ring of integers of $\tilde{\Omega}_0$. We extend this to a filtration on $\Omega_{C(b)}$ by defining

$$F^r \Omega_{C(b)}^k = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} F^r C(b) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

This filtration is exhaustive and separated, i. e.,

$$\bigcup_{r \in \mathbf{Z}} F^r \Omega_{C(b)} = \Omega_{C(b)} \quad \text{and} \quad \bigcap_{r \in \mathbf{Z}} F^r \Omega_{C(b)} = (0).$$

We normalize the D_i so that they respect this filtration. Put

$$\epsilon = Mb(d-1) - Md/(p-1),$$

a nonnegative integer. Then

$$\pi^\epsilon D_i(F^r C(b)) \subseteq F^r C(b)$$

and the complexes $(\Omega_{C(b)}, D)$, $(\Omega_{C(b)}, \pi^\epsilon D)$ have the same cohomology.

Since $(\Omega_{C(b)}, \pi^\epsilon D)$ is a filtered complex, there is an associated spectral sequence. Its E_1 -term is given by

$$E_1^{r,s} = H^{r+s}(F^r \Omega_{C(b)} / F^{r+1} \Omega_{C(b)}).$$

Consider the map $F^0 C(b) \rightarrow \mathbf{F}_q[x_1, \dots, x_n]$ defined by

$$\sum_u A_u \pi^{Mb|u|} x^u \mapsto \sum_u \bar{A}_u x^u,$$

where \bar{A}_u denotes the reduction of A_u modulo the maximal ideal of $\mathcal{O}_{\tilde{\Omega}_0}$. (Since $A_u \rightarrow 0$ as $u \rightarrow \infty$, the sum on the right-hand side is finite.) This map induces an isomorphism

$$(3.1) \quad F^0 \Omega_{C(b)}^k / F^1 \Omega_{C(b)}^k \simeq \Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}^k.$$

In particular,

$$(3.2) \quad F^0 C(b) / F^1 C(b) \simeq \mathbf{F}_q[x_1, \dots, x_n].$$

We have clearly

$$\frac{\partial}{\partial x_i}(F^r C(b)) \subseteq F^{r+1} C(b),$$

and a calculation show that

$$\begin{aligned} \pi^\epsilon H_i &\equiv \pi^{Mb(d-1)} \hat{f}_i \pmod{F^1 C(b)} \\ &\equiv \pi^{Mb(d-1)} \hat{f}_i^{(d)} \pmod{F^1 C(b)}, \end{aligned}$$

hence under the isomorphism (3.2), the map

$$\pi^\epsilon D_i : F^0 C(b) \rightarrow F^0 C(b)$$

induces the map “multiplication by $\partial f^{(d)}/\partial x_i$ ” on $\mathbf{F}_q[x_1, \dots, x_n]$. More generally, one sees that under the isomorphism (3.1), the map

$$\pi^\epsilon D : F^0 \Omega_{C(b)}^k \rightarrow F^0 \Omega_{C(b)}^{k+1}$$

induces the map

$$\phi_{f^{(d)}} : \Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}^k \rightarrow \Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}^{k+1}$$

defined by

$$\phi_{f^{(d)}}(\omega) = df^{(d)} \wedge \omega,$$

where $df^{(d)}$ denotes the exterior derivative of $f^{(d)}$. We have proved that there is an isomorphism of complexes of \mathbf{F}_q -vector spaces

$$(F^0 \Omega_{C(b)}/F^1 \Omega_{C(b)}, \pi^\epsilon D) \simeq (\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}}).$$

Since multiplication by π^r defines an isomorphism of complexes

$$(F^0 \Omega_{C(b)}, \pi^\epsilon D) \simeq (F^r \Omega_{C(b)}, \pi^\epsilon D),$$

we have in fact isomorphisms for all $r \in \mathbf{Z}$

$$(3.3) \quad (F^r \Omega_{C(b)}/F^{r+1} \Omega_{C(b)}, \pi^\epsilon D) \simeq (\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}}).$$

The complex $(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}})$ is isomorphic to the Koszul complex on $\mathbf{F}_q[x_1, \dots, x_n]$ defined by $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$. If we assume $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$, we get

$$(3.4) \quad H^i(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}}) = 0 \quad \text{for } i \neq n,$$

$$(3.5) \quad \dim_{\mathbf{F}_q} H^n(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}}) = (d-1)^n.$$

It follows from these equations that

$$(3.6) \quad E_1^{r,s} = 0 \quad \text{if } r+s \neq n$$

$$(3.7) \quad \dim_{\mathbf{F}_q} E_1^{r,s} = (d-1)^n \quad \text{if } r+s = n.$$

The first of these equalities implies that all the coboundary maps $d_1^{r,s}$ are zero, hence the spectral sequence converges weakly, i. e.,

$$E_1^{r,s} \simeq F^r H^{r+s}(\Omega_{C(b)}, \pi^\epsilon D)/F^{r+1} H^{r+s}(\Omega_{C(b)}, \pi^\epsilon D).$$

This spectral sequence actually converges. First observe the following. Let x^{μ_i} , $i = 1, \dots, (d-1)^n$, be monomials in x_1, \dots, x_n such that the cohomology classes $\{[x^{\mu_i} dx_1 \wedge \dots \wedge dx_n]\}_{i=1}^{(d-1)^n}$ form a basis for $H^n(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}})$. Then the

images of the cohomology classes $\{\pi^r x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n\}_{i=1}^{(d-1)^n}$ in $E_1^{r,s}$ form a basis for $E_1^{r,s}$ when $r + s = n$.

Theorem 3.8. *Suppose $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$. Then*

- (1) $H^i(\Omega_{C(b)}, \pi^\epsilon D) = 0$ if $i \neq n$,
- (2) *The cohomology classes $[x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n]$, $i = 1, \dots, (d-1)^n$, form a basis for $H^n(\Omega_{C(b)}, \pi^\epsilon D)$.*

Proof. Suppose $i \neq n$ and let $\eta \in \Omega_{C(b)}^i$ with $\pi^\epsilon D(\eta) = 0$. For some r we have $\eta \in F^r \Omega_{C(b)}^i$. Equations (3.3) and (3.4) then imply that

$$\eta = \pi \eta_1 + \pi^\epsilon D(\zeta_1)$$

with $\eta_1 \in F^r \Omega_{C(b)}^i$ and $\zeta_1 \in F^r \Omega_{C(b)}^{i-1}$. Suppose that for some $t \geq 1$ we have found $\eta_t \in F^r \Omega_{C(b)}^i$ and $\zeta_t \in F^r \Omega_{C(b)}^{i-1}$ such that

$$(3.9) \quad \eta = \pi^t \eta_t + \pi^\epsilon D(\zeta_t)$$

and such that

$$\zeta_t - \zeta_{t-1} \in F^{r+t-1} \Omega_{C(b)}^{i-1}.$$

Applying $\pi^\epsilon D$ to both sides of (3.9) gives

$$\pi^{t+\epsilon} D(\eta_t) = 0,$$

hence $\pi^\epsilon D(\eta_t) = 0$ since multiplication by π is injective on $\Omega_{C(b)}^i$. Equations (3.3) and (3.4) give

$$\eta_t = \pi \eta_{t+1} + \pi^\epsilon D(\zeta'_{t+1}),$$

with $\eta_{t+1} \in F^r \Omega_{C(b)}^i$ and $\zeta'_{t+1} \in F^r \Omega_{C(b)}^{i-1}$. If we put $\zeta_{t+1} = \zeta_t + \pi^t \zeta'_{t+1}$, then substitution into (3.9) gives

$$\eta = \pi^{t+1} \eta_{t+1} + \pi^\epsilon D(\zeta_{t+1})$$

with

$$\zeta_{t+1} - \zeta_t \in F^{r+t} \Omega_{C(b)}^{i-1}.$$

It is now clear that the sequence $\{\zeta_t\}_{t=1}^\infty$ converges to an element $\zeta \in F^r \Omega_{C(b)}^{i-1}$ such that $\eta = \pi^\epsilon D(\zeta)$. This proves the first assertion.

It follows easily from (3.3) that the $\{[x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n]\}_{i=1}^{(d-1)^n}$ are linearly independent, hence it suffices to show that they span $H^n(\Omega_{C(b)}, \pi^\epsilon D)$. Let $\eta \in F^r \Omega_{C(b)}^n$. From (3.3) we have

$$\eta = \sum_{i=1}^{(d-1)^n} c_i^{(1)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi \eta_1 + \pi^\epsilon D(\zeta_1),$$

where $c_i^{(1)} \in \tilde{\Omega}_0$, $c_i^{(1)} x^{\mu_i} \in F^r C(b)$, $\eta_1 \in F^r \Omega_{C(b)}^n$, $\zeta_1 \in F^r \Omega_{C(b)}^{n-1}$. Suppose we can write

$$(3.10) \quad \eta = \sum_{i=1}^{(d-1)^n} c_i^{(t)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^t \eta_t + \pi^\epsilon D(\zeta_t)$$

with $c_i^{(t)} \in \tilde{\Omega}_0$, $c_i^{(t)} x^{\mu_i} \in F^r C(b)$, $\eta_t \in F^r \Omega_{C(b)}^n$, and $\zeta_t \in F^r \Omega_{C(b)}^{n-1}$ such that

$$\begin{aligned} (c_i^{(t)} - c_i^{(t-1)}) x^{\mu_i} &\in F^{r+t-1} C(b) \\ \zeta_t - \zeta_{t-1} &\in F^{r+t-1} \Omega_{C(b)}^{n-1}. \end{aligned}$$

By (3.3) we have

$$\eta_t = \sum_{i=1}^{(d-1)^n} c'_i x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi \eta_{t+1} + \pi^\epsilon D(\zeta'_t),$$

where $c'_i \in \tilde{\Omega}_0$, $c'_i x^{\mu_i} \in F^r C(b)$, $\eta_{t+1} \in F^r \Omega_{C(b)}^n$, $\zeta'_t \in F^r \Omega_{C(b)}^{n-1}$. If we put $c_i^{(t+1)} = c_i^{(t)} + \pi^t c'_i$ and $\zeta_{t+1} = \zeta_t + \pi^t \zeta'_t$, then

$$\eta = \sum_{i=1}^{(d-1)^n} c_i^{(t+1)} x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^{t+1} \eta_{t+1} + \pi^\epsilon D(\zeta_{t+1})$$

with

$$\begin{aligned} (c_i^{(t+1)} - c_i^{(t)}) x^{\mu_i} &\in F^{r+t} C(b) \\ \zeta_{t+1} - \zeta_t &\in F^{r+t} \Omega_{C(b)}^{n-1}. \end{aligned}$$

It follows that the sequences $\{c_i^{(t)}\}_{t=1}^\infty$ and $\{\zeta_t\}_{t=1}^\infty$ converge, say, $c_i^{(t)} \rightarrow c_i \in \tilde{\Omega}_0$, $\zeta_t \rightarrow \zeta \in F^r \Omega_{C(b)}^{n-1}$, and that these limits satisfy

$$\eta = \sum_{i=1}^{(d-1)^n} c_i x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n + \pi^\epsilon D(\zeta)$$

with $c_i x^{\mu_i} \in F^r C(b)$. This completes the proof of the second assertion.

The following result is a consequence of the proof of Theorem 3.8.

Proposition 3.11. *Under the hypothesis of Theorem 3.8, if $\eta \in F^r \Omega_{C(b)}^n$, then there exist $\{c_i\}_{i=1}^{(d-1)^n} \subseteq \tilde{\Omega}_0$ such that in $H^n(\Omega_{C(b)}, \pi^\epsilon D)$ we have*

$$[\eta] = \sum_{i=1}^{(d-1)^n} [c_i x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n],$$

where $c_i x^{\mu_i} \in F^r C(b)$ for $i = 1, \dots, (d-1)^n$.

4. p -ADIC ESTIMATES

It follows from (2.14) and Theorem 3.8 that

$$(4.1) \quad L(\mathbf{A}^n, f; t)^{(-1)^{n+1}} = \det(I - t\alpha_n \mid H^n(\Omega_{C(b)}, D))$$

is a polynomial of degree $(d-1)^n$ (by [8], zero is not an eigenvalue of α_n). We estimate its p -adic Newton polygon. Note that

$$H^n(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}}) \simeq \mathbf{F}_q[x_1, \dots, x_n] / (\partial f^{(d)} / \partial x_1, \dots, \partial f^{(d)} / \partial x_n)$$

is a graded $\mathbf{F}_q[x_1, \dots, x_n]$ -module. Let $H^n(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}, \phi_{f^{(d)}})^{(m)}$ denote its homogeneous component of degree m . It follows from (3.4) that its Hilbert-Poincare

series is $(1 + t + \cdots + t^{d-2})^n$. Write

$$(4.2) \quad (1 + t + \cdots + t^{d-2})^n = \sum_{m=0}^{n(d-2)} U_m t^m,$$

so that

$$U_m = \dim_{\mathbf{F}_q} H^n(\Omega_{\mathbf{F}_q[x_1, \dots, x_n]/\mathbf{F}_q}^n, \phi_{f^{(d)}}^{(m)}).$$

Equivalently,

$$U_m = \text{card}\{x^{\mu_i} \mid |\mu_i| = m\}.$$

Theorem 4.3. *Suppose $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$. Then the Newton polygon of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ with respect to the valuation “ord_q” lies on or above the Newton polygon with respect to the valuation “ord_q” of the polynomial*

$$\prod_{m=0}^{n(d-2)} (1 - q^{(m+n)/d} t)^{U_m}.$$

We begin with a reduction step. Let β_n be the endomorphism of $H^n(\Omega_{C(b)}^n, D)$ constructed from β as α_n was constructed from α , i. e.,

$$\beta_n(\xi dx_1 \wedge \cdots \wedge dx_n) = \frac{1}{x_1 \cdots x_n} \beta(x_1 \cdots x_n \xi) dx_1 \wedge \cdots \wedge dx_n.$$

Then β_n is an $\tilde{\Omega}_1$ -linear endomorphism of $H^n(\Omega_{C(b)}^n, D)$ and $\alpha_n = (\beta_n)^a$. By [5, Lemma 7.1], we have the following.

Lemma 4.4. *The Newton polygon of $\det_{\tilde{\Omega}_0}(I - t\alpha_n \mid H^n(\Omega_{C(b)}^n, D))$ with respect to the valuation “ord_q” is obtained from the Newton polygon of $\det_{\tilde{\Omega}_1}(I - t\beta_n \mid H^n(\Omega_{C(b)}^n, D))$ with respect to the valuation “ord” by shrinking ordinates and abscissas by a factor of $1/a$.*

Let $\{\gamma_j\}_{j=1}^a$ be an integral basis for $\tilde{\Omega}_0$ over $\tilde{\Omega}_1$. Then under the hypothesis of Theorem 3.8, the cohomology classes

$$[\gamma_j x^{\mu_i} dx_1 \wedge \cdots \wedge dx_n], \quad i = 1, \dots, (d-1)^n, j = 1, \dots, a,$$

form a basis for $H^n(\Omega_{C(b)}^n, D)$ as $\tilde{\Omega}_1$ -vector space. We estimate p -adically the entries of the matrix of β_n with respect to a certain normalization of this basis, namely, we set

$$\xi(i, j) = (\pi^{Mb/p})^{|\mu_i|+n} \gamma_j x^{\mu_i}$$

and use the cohomology classes $[\xi(i, j) dx_1 \wedge \cdots \wedge dx_n]$. This normalization is chosen so that

$$x_1 \cdots x_n \xi(i, j) \in C(b/p, 0),$$

hence

$$\beta(x_1 \cdots x_n \xi(i, j)) \in C(b, 0)$$

and

$$\frac{1}{x_1 \cdots x_n} \beta(x_1 \cdots x_n \xi(i, j)) \in \pi^{Mbn} C(b, 0).$$

This says that

$$\beta_n(\xi(i, j) dx_1 \wedge \cdots \wedge dx_n) \in F^{Mbn} \Omega_{C(b)}^n,$$

hence by Proposition 3.11 and the properties of an integral basis we have

$$[\beta_n(\xi(i, j) dx_1 \wedge \cdots \wedge dx_n)] = \sum_{i', j'} A(i', j'; i, j) [\gamma_{j'} x^{\mu_{i'}} dx_1 \wedge \cdots \wedge dx_n]$$

with $A(i', j'; i, j) \gamma_{j'} x^{\mu_{i'}} \in F^{Mbn} C(b)$, i. e., $A(i', j'; i, j) \in \pi^{Mb(|\mu_i|+n)} \mathcal{O}_{\tilde{\Omega}_0}$. This may be rewritten as

$$[\beta_n(\xi(i, j) dx_1 \wedge \cdots \wedge dx_n)] = \sum_{i', j'} B(i', j'; i, j) [\xi(i', j') dx_1 \wedge \cdots \wedge dx_n]$$

with

$$B(i', j'; i, j) \in \pi^{Mb(|\mu_{i'}|+n)(1-1/p)} \mathcal{O}_{\tilde{\Omega}_0},$$

i. e., the (i', j') -row of the matrix $B(i', j'; i, j)$ of β_n with respect to the basis $\{[\xi(i, j) dx_1 \wedge \cdots \wedge dx_n]\}_{i, j}$ is divisible by

$$\pi^{Mb(|\mu_{i'}|+n)(1-1/p)}.$$

This implies that $\det_{\tilde{\Omega}_1}(I - t\beta_n \mid H^n(\Omega_{C(b)}, D))$ has Newton polygon (with respect to the valuation “ord”) lying on or above the Newton polygon (with respect to the valuation “ord”) of the polynomial

$$\prod_{m=0}^{n(d-2)} (1 - \pi^{Mb(m+n)(1-1/p)} t)^{aU_m}.$$

But $\det_{\tilde{\Omega}_1}(I - t\beta_n \mid H^n(\Omega_{C(b)}, D))$ is independent of b , so we may take the limit as $b \rightarrow p/(p-1)$ to conclude that its Newton polygon lies on or above the Newton polygon of

$$\prod_{m=0}^{n(d-2)} (1 - p^{(m+n)/d} t)^{aU_m}.$$

Theorem 4.3 now follows from Lemma 4.4.

Let $\{\rho_i\}_{i=1}^{(d-1)^n}$ be the reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ and put

$$\Lambda(f) = \prod_{i=1}^{(d-1)^n} \rho_i \in \mathbf{Q}(\zeta_p).$$

Theorem 4.3 implies that

$$\text{ord}_q \Lambda(f) \geq \frac{1}{d} \sum_{m=0}^{n(d-2)} (m+n) U_m.$$

But it follows from (4.2) evaluated at $t = 1$ that

$$\sum_{m=0}^{n(d-2)} U_m = (d-1)^n$$

and from the derivative of (4.2) evaluated at $t = 1$ that

$$\sum_{m=0}^{n(d-2)} m U_m = n(d-1)^n(d-2)/2.$$

We thus get the following.

Corollary 4.5. *Under the hypothesis of Theorem 4.3,*

$$\mathrm{ord}_q \Lambda(f) \geq \frac{n(d-1)^n}{2}.$$

It can be proved directly by p -adic methods that equality holds in Corollary 4.5. We shall derive this equality in the next section by l -adic methods.

5. l -ADIC COHOMOLOGY

Let l be a prime, $l \neq p$. There exists a lisse, rank-one, l -adic étale sheaf $\mathcal{L}_\Psi(f)$ on \mathbf{A}^n with the property that

$$(5.1) \quad L(\mathbf{A}^n, f; t) = L(\mathbf{A}^n, \mathcal{L}_\Psi(f); t),$$

where the right-hand side is a Grothendieck L -function. By Grothendieck's Lefschetz trace formula,

$$(5.2) \quad L(\mathbf{A}^n, f; t) = \prod_{i=0}^{2n} \det(I - tF \mid H_c^i(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)))^{(-1)^{i+1}},$$

where H_c^i denotes l -adic cohomology with proper supports and F is the Frobenius endomorphism. The $H_c^i(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f))$ are finite-dimensional vector spaces over a finite extension K_l of \mathbf{Q}_l containing the p -th roots of unity. We combine Theorem 3.8 and Corollary 4.5 with general results of Deligne[3] to prove the following theorem of Deligne[2, Théorème 8.4].

Theorem 5.3. *Suppose $(p, d) = 1$ and $f^{(d)} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-1} . Then*

- (1) $H_c^i(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = 0$ if $i \neq n$,
- (2) $\dim_{K_l} H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = (d-1)^n$,
- (3) $H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f))$ is pure of weight n .

Proof. We consider the theorem to be known for $n = 1$ and prove it for general $n \geq 2$ by induction. For $\lambda \in \bar{\mathbf{F}}_q$, set

$$f_\lambda(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, \lambda) \in \mathbf{F}_q(\lambda)[x_1, \dots, x_{n-1}].$$

Since the generic hyperplane section of a smooth variety is smooth, we may assume, after a coordinate change if necessary, that the hyperplane $x_n = 0$ intersects the variety $f^{(d)} = 0$ transversally in \mathbf{P}^{n-1} . Thus $f^{(d)}(x_1, \dots, x_{n-1}, 0) = 0$ defines a smooth hypersurface in \mathbf{P}^{n-2} . But

$$f_\lambda^{(d)} = f^{(d)}(x_1, \dots, x_{n-1}, 0),$$

so by the induction hypothesis the conclusions of the theorem are true for all f_λ .

Consider the morphism of \mathbf{F}_q -schemes $\sigma : \mathbf{A}^n \rightarrow \mathbf{A}^1$ which is projection onto the n -th coordinate. The Leray spectral sequence for the composition of σ with the structural morphism $\mathbf{A}^1 \rightarrow \mathrm{Spec}(\mathbf{F}_q)$ is

$$(5.4) \quad H_c^i(\mathbf{A}^1 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, R^j \sigma_!(\mathcal{L}_\Psi(f))) \Rightarrow H_c^{i+j}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)).$$

Proper base change implies that for $\lambda \in \bar{\mathbf{F}}_q$ and $\bar{\lambda}$ a geometric point over λ

$$(5.5) \quad (R^j \sigma_!(\mathcal{L}_\Psi(f)))_{\bar{\lambda}} = H_c^j(\sigma^{-1}(\lambda) \times_{\mathbf{F}_q(\lambda)} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f_\lambda)).$$

Applying the induction hypothesis to f_λ shows that the right-hand side of (5.5) vanishes for all $\lambda \in \bar{\mathbf{F}}_q$ if $j \neq n-1$. It follows that the Leray spectral sequence collapses and we get

$$(5.6) \quad H_c^i(\mathbf{A}^1 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, R^{n-1}\sigma_!(\mathcal{L}_\Psi(f))) = H_c^{i+n-1}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)).$$

Since $\dim \mathbf{A}^1 = 1$, the left-hand side of (5.6) can be nonzero only for $i = 0, 1, 2$. However, $H_c^{n-1}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = 0$ because \mathbf{A}^n is smooth, affine, of dimension n , and $\mathcal{L}_\Psi(f)$ is lisse on \mathbf{A}^n . This proves that $H_c^i(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = 0$ except possibly for $i = n, n+1$.

By (5.2) we then have

$$(5.7) \quad L(\mathbf{A}^n, f; t)^{(-1)^{n+1}} = \frac{\det(I - tF \mid H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)))}{\det(I - tF \mid H_c^{n+1}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)))}.$$

Since $\mathcal{L}_\Psi(f)$ is pure of weight 0, Deligne's fundamental theorem[3] tells us that $H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f))$ is mixed of weights $\leq n$. Equation (5.5) and the induction hypothesis applied to f_λ tell us that $R^{n-1}\sigma_!(\mathcal{L}_\Psi(f))$ is pure of weight $n-1$ and that all fibers of $R^{n-1}\sigma_!(\mathcal{L}_\Psi(f))$ have the same rank, namely, $(d-1)^{n-1}$. It follows from Katz[7, Corollary 6.7.2] that $R^{n-1}\sigma_!(\mathcal{L}_\Psi(f))$ is lisse on \mathbf{A}^1 . Equation (5.6) with $i = 2$ now implies, by Deligne[3, Corollaire 1.4.3], that $H_c^{n+1}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f))$ is pure of weight $n+1$, hence there can be no cancellation on the right-hand side of (5.7). However, Theorem 3.8 implies that $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n$, so we must have

$$H_c^{n+1}(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = 0$$

and

$$\dim_{K_i} H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f)) = (d-1)^n.$$

This establishes the first two assertions of the theorem.

To prove the last assertion of the theorem, note that $|\rho_i| \leq q^{n/2}$ for every i and every archimedean absolute value since $H_c^n(\mathbf{A}^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{L}_\Psi(f))$ is mixed of weights $\leq n$. Thus we have

$$(5.8) \quad |\Lambda(f)| \leq q^{n(d-1)^n/2}$$

for every archimedean absolute value on $\mathbf{Q}(\zeta_p)$. By Corollary 4.5, we have

$$(5.9) \quad |\Lambda(f)|_p \leq q^{-n(d-1)^n/2}$$

for every normalized archimedean absolute value on $\mathbf{Q}(\zeta_p)$ lying over p , and it is well-known that $|\rho_i|_{p'} = 1$ for every nonarchimedean absolute value lying over any prime $p' \neq p$. It then follows from the product formula for $\mathbf{Q}(\zeta_p)$ that equality holds in (5.8) (and also in (5.9)), which implies the last assertion of the theorem.

6. PROOF OF THEOREM 1.4

It remains to consider the case where p divides d . The Euler relation becomes

$$\sum_{i=1}^n x_i \frac{\partial f^{(d)}}{\partial x_i} = 0.$$

The regular sequence hypothesis then implies that

$$x_i \in \left(\frac{\partial f^{(d)}}{\partial x_1}, \dots, \widehat{\frac{\partial f^{(d)}}{\partial x_i}}, \dots, \frac{\partial f^{(d)}}{\partial x_n} \right),$$

hence there is an equality of ideals of $\mathbf{F}_q[x_1, \dots, x_n]$

$$(6.1) \quad (x_1, \dots, x_n) = \left(\frac{\partial f^{(d)}}{\partial x_1}, \dots, \frac{\partial f^{(d)}}{\partial x_n} \right).$$

Conversely, if (6.1) holds, then $\{\partial f^{(d)}/\partial x_i\}_{i=1}^n$ is a regular sequence. Equation (6.1) implies that $d = 2$, hence $p = 2$ as well, thus f is a quadratic polynomial in characteristic 2. We may assume f contains no terms of the form x_i^2 by the following elementary lemma.

Let ζ_p be a primitive p -th root of unity. Since Ψ is a nontrivial additive character of \mathbf{F}_q , there exists a nonzero $b \in \mathbf{F}_q$ such that

$$(6.2) \quad \Psi(x) = \zeta_p^{\text{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(bx)}.$$

Lemma 6.3. *Let $a \in \mathbf{F}_q$, $a \neq 0$, and choose $c \in \mathbf{F}_q$ such that $c^p = (ab)^{-1}$. Then*

$$\sum_{x_1, \dots, x_n \in \mathbf{F}_q} \Psi(f(x_1, \dots, x_n) + ax_n^p) = \sum_{x_1, \dots, x_n \in \mathbf{F}_q} \Psi(f(x_1, \dots, x_n) + ac^{p-1}x_n).$$

Proof. Making the change of variable $x_n \mapsto cx_n$, the sum becomes

$$\begin{aligned} \sum_{x_1, \dots, x_n \in \mathbf{F}_q} \Psi(f(x_1, \dots, x_{n-1}, cx_n) + b^{-1}x_n^p) = \\ \sum_{x_1, \dots, x_n \in \mathbf{F}_q} \Psi(f(x_1, \dots, x_{n-1}, cx_n) + b^{-1}x_n) \Psi(b^{-1}(x_n^p - x_n)). \end{aligned}$$

But by (6.2),

$$\begin{aligned} \Psi(b^{-1}(x_n^p - x_n)) &= \zeta_p^{\text{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_n^p - x_n)} \\ &= 1 \end{aligned}$$

since $\text{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_n^p - x_n) = 0$ for all $x_n \in \mathbf{F}_q$. Making the change of variable $x_n \mapsto c^{-1}x_n$ now gives the lemma.

By the lemma, we may assume our quadratic polynomial f has the form

$$f = \sum_{1 \leq i < j \leq n} a_{ij}x_i x_j + \sum_{k=1}^n b_k x_k + c,$$

where $a_{ij}, b_k, c \in \mathbf{F}_q$. This gives

$$f^{(2)} = \sum_{1 \leq i < j \leq n} a_{ij}x_i x_j.$$

Let $A = (A_{ij})$ be the $n \times n$ matrix defined by

$$A_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ a_{ji} & \text{if } i > j. \end{cases}$$

Thus A is a symmetric matrix with zeros on the diagonal. One checks that

$$\begin{bmatrix} \frac{\partial f^{(2)}}{\partial x_1} \\ \dots \\ \frac{\partial f^{(2)}}{\partial x_n} \end{bmatrix} = A \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix},$$

therefore (6.1) holds if and only if $\det A \neq 0$.

We now evaluate the exponential sum

$$(6.4) \quad \sum_{x_1, \dots, x_n \in \mathbf{F}_q} \Psi \left(\sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{k=1}^n b_k x_k + c \right).$$

Proposition 6.5. *If n is odd, then (6.1) cannot hold. If n is even and (6.1) holds, then the sum (6.4) equals $\zeta q^{n/2}$, where ζ is a root of unity.*

Proof. If $n = 1$, then $\det A = 0$, so (6.1) cannot hold. If $n = 2$, then $\det A \neq 0$ if and only if $a_{12} \neq 0$. It is then easy to check that the sum (6.4) equals

$$\Psi \left(\frac{b_1 b_2}{a_{12}} + c \right) q.$$

Thus the proposition holds for $n = 1, 2$. Suppose $n \geq 3$. The sum (6.4) can be rewritten as

$$\sum_{x_1, \dots, x_{n-1} \in \mathbf{F}_q} \Psi \left(\sum_{1 \leq i < j \leq n-1} a_{ij} x_i x_j + \sum_{k=1}^{n-1} b_k x_k + c \right) \sum_{x_n \in \mathbf{F}_q} \Psi \left(\left(\sum_{i=1}^{n-1} a_{in} x_i + b_n \right) x_n \right).$$

But

$$\sum_{x_n \in \mathbf{F}_q} \Psi \left(\left(\sum_{i=1}^{n-1} a_{in} x_i + b_n \right) x_n \right) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n-1} a_{in} x_i + b_n \neq 0 \\ q & \text{if } \sum_{i=1}^{n-1} a_{in} x_i + b_n = 0, \end{cases}$$

hence (6.4) equals

$$(6.6) \quad q \sum_{\substack{x_1, \dots, x_{n-1} \in \mathbf{F}_q \\ \sum_{i=1}^{n-1} a_{in} x_i + b_n = 0}} \Psi \left(\sum_{1 \leq i < j \leq n-1} a_{ij} x_i x_j + \sum_{k=1}^{n-1} b_k x_k + c \right).$$

Since we are assuming A is invertible, some a_{in} must be nonzero, say, $a_{n-1,n} \neq 0$. By making the change of variable $x_{n-1} \mapsto (a_{n-1,n})^{-1} x_{n-1}$, we may assume $a_{n-1,n} = 1$. Solving $a_{1n} x_1 + \dots + a_{n-1,n} x_{n-1} + b_n = 0$ for x_{n-1} and substituting into the expression in the additive character, we see that (6.6) equals

$$(6.7) \quad q \sum_{x_1, \dots, x_{n-2} \in \mathbf{F}_q} \Psi \left(\sum_{1 \leq i < j \leq n-2} a'_{ij} x_i x_j + \sum_{k=1}^{n-2} b'_k x_k + c \right),$$

where

$$a'_{ij} = a_{ij} + a_{i,n-1} a_{jn} + a_{j,n-1} a_{in}.$$

Let $A' = (A'_{ij})$ be the $(n-2) \times (n-2)$ matrix constructed from the a'_{ij} as A was constructed from the a_{ij} . We explain the connection between A and A' . Let \tilde{A} be the $n \times n$ matrix obtained from A by replacing row i by

$$\text{row } i + a_{in}(\text{row } n-1) + a_{i,n-1}(\text{row } n)$$

for $i = 1, \dots, n-2$. Keeping in mind that $a_{n-1,n} = 1$, we see that

$$\tilde{A} = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline a_{1,n-1} & \cdots & a_{n-2,n-1} & 0 & 1 \\ a_{1n} & \cdots & a_{n-2,n} & 1 & 0 \end{array} \right].$$

In particular, $\det A' = \det A$. We can repeat this procedure starting with the sum (6.7) and continue until we are reduced to the one or two variable case, according to whether n is odd or even. If n is odd, this implies $\det A = 0$, a contradiction. Thus there does not exist a quadratic polynomial f satisfying (6.1) in this case. If n is even, this shows that (6.4) equals $q^{n/2}$ times a root of unity, which is the desired result.

A straightforward calculation using Proposition 6.5 then shows that the corresponding L -function has the form asserted in Theorem 1.4.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078
E-mail address: `adolphs@math.okstate.edu`

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455
E-mail address: `sperber@math.umn.edu`